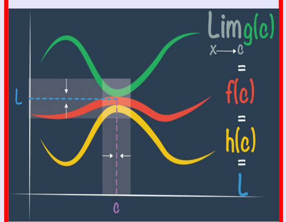
Math 261

Spring 2022

Lecture 22



Find $f(x)$

1) $f'(x) = x^2 - x^{-2}$

$$f(x) = \frac{x^{2+1}}{2+1} - \frac{x^{-2+1}}{-2+1} + C$$

$$f(x) = \frac{x^3}{3} - \frac{x^{-1}}{-1} + C$$

$$\boxed{f(x) = \frac{1}{3}x^3 + \frac{1}{x} + C}$$

2) $f'(x) = 8x - 3\sec^2 x$

$$f(x) = \frac{8x^2}{2} - 3\cdot\tan x + C$$

$$\boxed{f(x) = 4x^2 - 3\tan x + C}$$

3) $f'(x) = \underbrace{\sec x \tan x} - \csc x \cot x$

$$f(x) = \sec x - (-\csc x) + C$$

$$\boxed{f(x) = \sec x + \csc x + C}$$

Find $f(x)$ where

$f''(x) = \sin x + 3\cos x$, $f(0)=0$, $f'(0)=2$

$f'(x) = -\cos x + 3\sin x + C$

$f'(0) = -\cancel{\cos 0}^{1} + 3\cancel{\sin 0}^{0} + C = 2$

$\boxed{C=3}$

$f'(x) = -\cos x + 3\sin x + 3$

$f(x) = -\sin x + 3\cdot(-\cos x) + 3x + C$

$f(x) = -\sin x - 3\cos x + 3x + C$

$f(0) = -\cancel{\sin 0}^{0} - 3\cancel{\cos 0}^{1} + \cancel{3(0)}^{0} + C = 0$

$\boxed{C=3}$

$\boxed{f(x) = -\sin x - 3\cos x + 3x + 3}$

Find $f(x)$ is

$f''(x) = 2x^3 + 3x^2 - 4x + 5$, $f(0)=2$, $f(1)=0$

$f'(x) = \frac{2x^4}{4} + \frac{3x^3}{3} - \frac{4x^2}{2} + 5x + C$

$f'(x) = \frac{1}{2}x^4 + x^3 - 2x^2 + 5x + C$

$f(x) = \frac{1}{2}\cdot\frac{x^5}{5} + \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + Cx + D$

$f(0) = 0 + 0 - 0 + 0 + 0 + D = 2$

$\boxed{D=2}$

$f(1) = \frac{1}{10} + \frac{1}{4} - \frac{2}{3} + \frac{5}{2} + C + 2 = 0$

$\boxed{C = ?}$

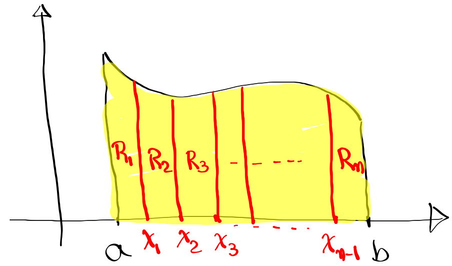
$\boxed{f(x) = \frac{1}{10}x^5 + \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 + Cx + 2}$

Suppose $f(x) \geq 0$ for all values on $[a,b]$.

Suppose $f(x)$ is continuous on $[a,b]$

Find the area below $f(x)$, above x-axis on $[a,b]$.

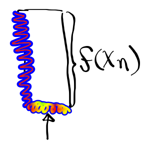
Dividing $[a,b]$ into n subintervals



$$A = \lim_{k \to \infty} \sum_{n=1}^{k} R_n$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \boxed{f(x_i)} \, \Delta x$$

Each region (think of small rectangle)



Delta x

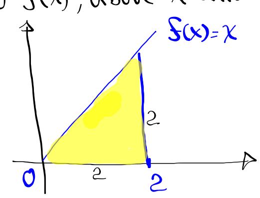
$$\Delta x = \frac{b-a}{n}$$

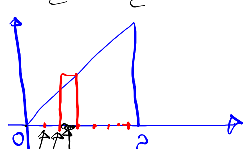
# of Subinterval

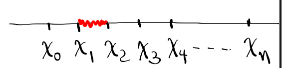
$f(x) = x$, $[0,2]$

Find the area below $f(x)$, above x-axis on $[0,2]$

$$A = \frac{bh}{2} = \frac{2 \cdot 2}{2} = \boxed{2}$$







$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

$x_0 = 0$

$x_1 = a + 1\Delta x = 1\Delta x$

$x_2 = a + 2\Delta x = 0 + 2\Delta x = 2\Delta x$

$x_3 = a + 3\Delta x = 0 + 3\Delta x = 3\Delta x$

$x_4 = \quad = 4\Delta x$

$\vdots$

$x_i = \quad = i\Delta x = \frac{2i}{n}$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \Delta x$$

$f(x) = x$

$f(x_i) = f\left(\frac{2i}{n}\right)$

$= \frac{2i}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2i}{n}\cdot\frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{4}{n^2} \sum_{i=1}^{n} i \rightarrow 1+2+3+\cdots+n$$

From PreCalc

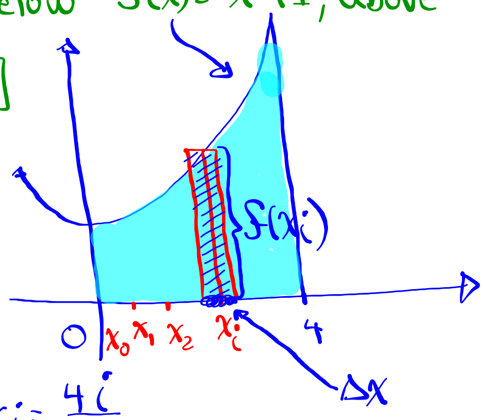
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{\overset{2}{\cancel{4}}}{n^2}\cdot\frac{n(n+1)}{\cancel{2}} = 2\lim_{n\to\infty}\frac{n^2+n}{n^2}$$

$$= 2\cdot\frac{1}{1}$$

$$= 2\cdot 1 = \boxed{2}$$

Find the area below $f(x) = x^2+1$, above x-axis, on $[0,4]$



$[a,b] = [0,4]$

$\Delta x = \frac{b-a}{n} = \frac{4}{n}$

$x_i = a + i\Delta x$

$= 0 + i\cdot\frac{4}{n}$ $x_i = \frac{4i}{n}$

$$f(x_i) = (x_i)^2+1 = \left(\frac{4i}{n}\right)^2+1 = \frac{16i^2}{n^2}+1$$

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{16i^2}{n^2}+1\right)\cdot\frac{4}{n}$$

$$A=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{16i^2}{n^2}+1\right)\cdot\frac{4}{n}$$

$$A=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{64i^2}{n^3}+\frac{4}{n}\right)$$

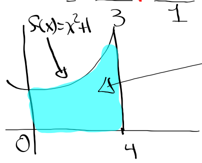
$$=\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{64i^2}{n^3}+\sum_{i=1}^{n}\frac{4}{n}\right]$$

$$=\lim_{n\to\infty}\left[\frac{64}{n^3}\cdot\sum_{i=1}^{n} i^2+\frac{4}{n}\underbrace{\sum_{i=1}^{n} 1}_{1+1+1+\dots+1\ n\text{ times}}\right]$$

$$=\lim_{n\to\infty}\left[\frac{\overset{32}{\cancel{64}}}{n^3}\cdot\frac{n(n+1)(2n+1)}{\cancel{6}_{3}}+\frac{4}{\cancel{n}}\cdot\cancel{n}\right]$$

$$=\frac{32}{3}\lim_{n\to\infty}\frac{\overset{2n^3+\cdots}{n(n+1)(2n+1)}}{n^3}+\lim_{n\to\infty}4$$

$$=\frac{32}{3}\cdot\frac{2}{1}+4=\frac{64}{3}+\frac{12}{3}=\boxed{\frac{76}{3}}$$



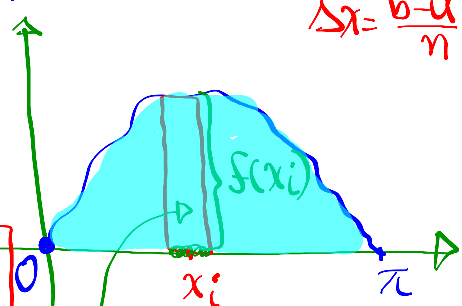
Find an expression for the area below $f(x)=\sqrt{\sin x}$, above x-axis for $0\le x\le\pi$.

$$\Delta x=\frac{b-a}{n}$$

$\sin x\ge 0$

$$x_i=a+i\,\Delta x$$

$$x_i=0+i\cdot\frac{\pi-0}{n}\to\boxed{x_i=\frac{\pi i}{n}}$$



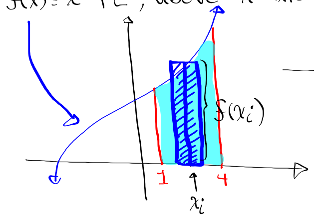
$$f(x_i)=\sqrt{\sin\left(\frac{\pi i}{n}\right)}$$

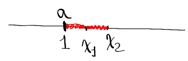
$$A_i=f(x_i)\cdot\Delta x$$

$$=\sqrt{\sin\left(\frac{\pi i}{n}\right)}\cdot\frac{\pi i}{n}$$

$$\boxed{A=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\sin\left(\frac{\pi i}{n}\right)}\cdot\frac{\pi i}{n}}\ \checkmark$$

Find an expression for the area below $f(x) = x^3 + 2$, above x-axis for $1 \le x \le 4$.





$x_0 = 1$
$x_1 = 1 + 1\Delta x$
$x_2 = 1 + 2\Delta x$
$x_3 = 1 + 3\Delta x$
$\vdots$
$x_i = 1 + i\Delta x$

$x_i = a + i\Delta x = 1 + i\Delta x$

$$\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$$

$$f(x_i) = (1 + i\Delta x)^3 + 2$$

$$A_i = f(x_i)\cdot\Delta x = \left[(1 + i\Delta x)^3 + 2\right]\cdot\frac{3}{n}$$

Area of the ith rectangle

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} A_i = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left((1+i\Delta x)^3 + 2\right)\frac{3}{n}\right]$$

Recall from PreCalc or College Algebra

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Indefinite Integral

$$\int f(x)\,dx = F(x) \iff F'(x) = f(x)$$

Integral | Integrand | Respect to x

Reverse of differentiation

$$\int \underbrace{2x}_{f(x)}\,dx = \underbrace{x^2 + C}_{F(x)} \quad \text{because} \quad \frac{d}{dx}\underbrace{[x^2 + C]}_{F(x)} = \underbrace{2x}_{f(x)}$$

$$\int \underbrace{\sec^2 x}_{f(x)}\,dx = \underbrace{\tan x + C}_{F(x)} \qquad \frac{d}{dx}[\tan x + C] = \sec^2 x$$

Find $\int \underbrace{(4x^3 - 6x)}_{f(x)}dx = \underbrace{\boxed{x^4 - 3x^2 + C}}_{F(x)}$

$$\frac{d}{dx}[x^4 - 3x^2 + C] = 4x^3 - 6x$$

Properties of Integrals:

$$\int c\, f(x)\,dx = c\int f(x)\,dx$$

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int k\,dx = kx + C \qquad \boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1}$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

Find $\int \frac{1}{x^4}\,dx$

$$\int \frac{1}{x^4}\,dx = \int x^{-4}\,dx = \frac{x^{-4+1}}{-4+1} + C$$

$$= \frac{x^{-3}}{-3} + C = \boxed{\frac{-1}{3x^3} + C}$$

Find $\int (x - 12\sin x + 8)\,dx$

$$= \int x\,dx - \int 12\sin x\,dx + \int 8\,dx =$$

$$= \frac{x^2}{2} - 12\cdot(-\cos x) + 8x + C$$

$$= \boxed{\frac{1}{2}x^2 + 12\cos x + 8x + C}$$

Find $\int\left(\sqrt{x^3}+\sqrt[3]{x^2}\right)dx$

$$=\int\left[x^{\frac{3}{2}}+x^{\frac{2}{3}}\right]dx$$

$$=\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}+\frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1}+C$$

Make Sure to Simplify

Find $\int\frac{\sin x+\sin x\cot^2 x}{\csc^2 x}\,dx$

$$=\int\frac{\sin x\cancel{\left[1+\cot^2 x\right]}}{\cancel{\csc^2 x}}\,dx$$

$$=\int\sin x\,dx=\boxed{-\cos x+C}$$

Find $\int(x^2+3)(x^2-3)\,dx=\int(x^4-9)\,dx$

$$=\boxed{\frac{x^5}{5}-9x+C}$$

Find $\int \sqrt{\frac{6}{x}}\, dx = \int \frac{\sqrt{6}}{\sqrt{x}} dx$

$$= \sqrt{6} \int x^{-1/2} dx = \sqrt{6} \cdot \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= \sqrt{6} \cdot \frac{\sqrt{x}}{\frac{1}{2}} + C$$

$$= 2\sqrt{6}\sqrt{x} + C$$

$$= \boxed{2\sqrt{6x} + C}$$

Find $\int \frac{1 + \sqrt[3]{x}}{\sqrt{x}} dx$

Algebra Hint:

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$= \int \frac{1}{\sqrt{x}} dx + \int \frac{\sqrt[3]{x}}{\sqrt{x}} dx$$

$$= \int x^{-1/2} dx + \int x^{\frac{1}{3}-\frac{1}{2}} dx = \int x^{-1/2} dx + \int x^{\frac{-1}{6}} dx$$

Use power rule to finish

Find $\int \frac{1 + \sin^2 x}{\sin^2 x}\,dx$

$$= \int \left(\frac{1}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x} \right) dx$$

$$= \int (\csc^2 x + 1)\,dx = \boxed{-\cot x + x + C}$$

Find

$$\int \left(x + \frac{1}{x}\right)^2 dx$$

Algebra Hint:
$x^2 = x \cdot x$
$(A+B)^2 =$
$A^2 + 2AB + B^2$

$$= \int \left(x^2 + 2 \cdot x \cdot \frac{1}{x} + \frac{1}{x^2}\right) dx$$

$$= \int (x^2 + 2 + x^{-2})\,dx =$$ Make Sure to finish

Definite Integral:

If $f(x)$ is a continuous function on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$ where $F'(x) = f(x)$

$$\int_0^2 x\,dx = \left.\frac{x^2}{2}\right|_0^2 = \frac{2^2}{2} - \frac{0^2}{2} = \frac{4}{2} - 0 = \boxed{2}$$

Evaluate

$$\int_0^2 (x^2+1)\,dx = \frac{x^3}{3} + x \Big|_0^2$$

$$= \left(\frac{2^3}{3} + 2\right) - \left(\cancel{\frac{0^3}{3} + 0}\right) \to 0$$

$$= \frac{8}{3} + 2 = \boxed{\frac{14}{3}}$$

Find $\int_1^4 (x^3+2)\,dx$

$$= \frac{x^4}{4} + 2x \Big|_1^4 = \left(\frac{4^4}{4} + 2(4)\right) - \left(\frac{1^4}{4} + 2(1)\right)$$

$$= \frac{256}{4} + 8 - \frac{1}{4} - 2$$

$$= \frac{255}{4} + 6 = \boxed{\frac{279}{4}}$$

Find $\int_0^{\pi} \cos x\,dx = \sin x \Big|_0^{\pi}$

$\lim_{n\to\infty} = 4$

$$= \cancel{\sin \pi} - \cancel{\sin 0} \quad (\to 0,\ \to 0)$$

$$= \boxed{0}$$

Evaluate $\int_0^{\pi/4} \sec^2 x\,dx = \tan x \Big|_0^{\pi/4}$

$$= \tan\frac{\pi}{4} - \tan 0 = \boxed{1}$$

$$\int (2x+1)^{10}\,dx$$

Let

$$u = 2x+1$$

$$\frac{du}{dx} = 2$$

$$du = 2\,dx$$

$$\frac{du}{2} = dx$$

$$= \int u^{10}\,\frac{du}{2}$$

$$= \frac{1}{2}\int u^{10}\,du$$

$$= \frac{1}{2}\cdot\frac{u^{11}}{11} + C = \frac{1}{22}u^{11} + C$$

Substitution method.

$$= \boxed{\frac{1}{22}(2x+1)^{11} + C}$$

Find $\int x \cos x^2 \, dx$

$$u = x^2$$

$$du = 2x\,dx$$

$$\frac{du}{2} = x\,dx$$

$$= \int \cos u \,\frac{du}{2} = \frac{1}{2}\int \cos u\,du$$

$$= \frac{1}{2}\cdot \sin u + C$$

$$= \boxed{\frac{1}{2}\sin x^2 + C}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$